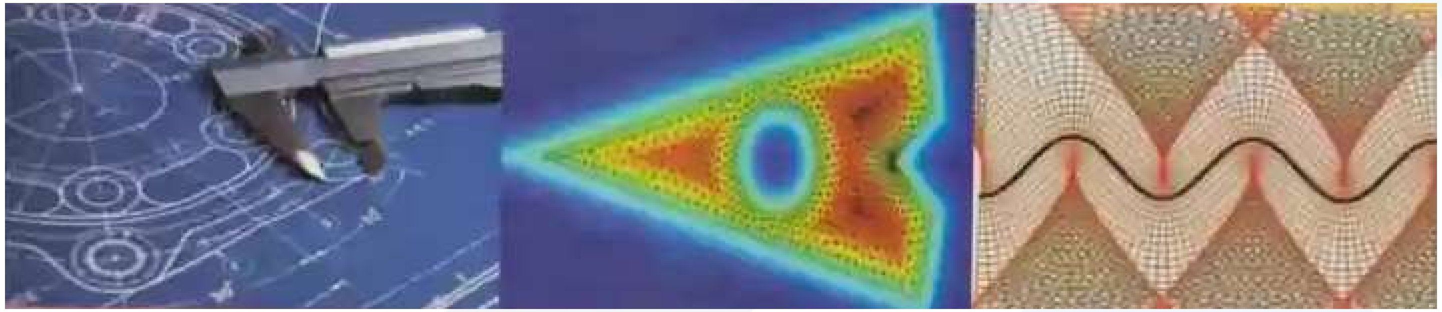
CEDC301: Mathematics Engineering

Exercises 4 & 5: Series and Residues



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1. Find the circle and radius of convergence of the given power series

$$\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{i}{1+i}\right)^k z^k \qquad \sum_{k=1}^{\infty} \frac{1}{k^2(3+4i)^k}(z+3i)^k \qquad \sum_{k=0}^{\infty} (-1)^k \left(\frac{1+2i}{2}\right)^k (z+2i)^k$$

$$\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}\left(\frac{i}{1+i}\right)^{n+1}}{\frac{1}{n}\left(\frac{i}{1+i}\right)^{n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}\left|\frac{i}{1+i}\right| = \frac{1}{\sqrt{2}} \Rightarrow R=\sqrt{2}, \quad |z|=\sqrt{2}$$

$$\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^2(3+4i)^{n+1}}}{\frac{1}{n^2(3+4i)^n}}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2\frac{1}{|3+4i|} = \frac{1}{5} \Rightarrow R=5, \quad |z+3i|=5$$

$$\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\left(\frac{1+2i}{2}\right)^n\right|}=\lim_{n\to\infty}\left|\frac{1+2i}{2}\right|=\frac{\sqrt{5}}{2}\Rightarrow R=\frac{2}{\sqrt{5}},\quad |z+2i|=\frac{2}{\sqrt{5}}$$

2. Show that the power series $\sum_{k=1}^{\infty} \frac{(z-i)^k}{k2^k}$ is not absolutely convergent on its circle of convergence. Determine at least one point on the circle of convergence at which the power series converges.

$$\lim_{n\to\infty}\left|\frac{k2^k}{(k+1)2^{k+1}}\right| = \frac{1}{2} \Rightarrow R = 2, \quad |z-i| = \sqrt{2}$$

$$\sum_{k=1}^{\infty}\left|\frac{(z-i)^k}{k2^k}\right| = \sum_{k=1}^{\infty}\frac{|(z-i)^k|}{k2^k} = \sum_{k=1}^{\infty}\frac{2^k}{k2^k} = \sum_{k=1}^{\infty}\frac{1}{k}$$ divergent harmonic series

$z = -2 + i$ is on the circle of convergence and $(z-i)^k = (-2)^k$

$$\sum_{k=1}^{\infty}\frac{(-2)^k}{k2^k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$ convergent

3. Expand the given function in a Maclaurin series

$$f(z) = \frac{i}{(z-i)(z-2i)} \qquad\qquad f(z) = \frac{z-7}{z^2-2z-3}$$

$$f(z) = \frac{1}{z-2i} - \frac{1}{z-i} = -\frac{1}{2i}\frac{1}{1-z/2i} + \frac{1}{2i}\frac{1}{1-z/i}$$

$$= -\frac{1}{2i}\left(1+\frac{z}{2i}+\frac{z^2}{(2i)^2}+\frac{z^3}{(2i)^3}+\cdots\right)+\frac{1}{2i}\left(1+\frac{z}{i}+\frac{z^2}{i^2}+\frac{z^3}{i^3}+\cdots\right)$$

$$= -\frac{i}{2} - \frac{3}{4}z + \frac{7i}{8}z^2 + \frac{15}{16}z^3 - \cdots$$

$$f(z)=\frac{2}{z+1}-\frac{1}{z-3}=2\frac{1}{1+z}+\frac{1}{3}\frac{1}{1-z/3}$$

$$=2\left(1-z+z^2-z^3+\cdots\right)+\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots\right)$$

$$=\frac{7}{3}-\frac{17}{9}z+\frac{55}{27}z^2-\frac{161}{81}z^3+\cdots$$

4. Expand the given function in a Taylor series centered at the indicated point. Give the radius of convergence of each series

$$f(z) = \frac{1}{1+z},\ z_0 = -i \qquad f(z) = \frac{z-1}{3-z},\ z_0 = 1 \qquad f(z) = \frac{1+z}{1-z},\ z_0 = i$$

$$\frac{1}{z+1} = \frac{1}{1-i+z+i} = \frac{1}{1-i}\,\frac{1}{1+\frac{z+i}{1-i}} = \frac{1}{1-i}\left[1 - \frac{z+i}{1-i} + \frac{(z+i)^2}{(1-i)^2} - \frac{(z+i)^3}{(1-i)^3} + \cdots\right]$$

$$= \frac{1}{1-i} - \frac{z+i}{(1-i)^2} + \frac{(z+i)^2}{(1-i)^3} - \frac{(z+i)^3}{(1-i)^4} + \cdots = \sum_{k=0}^{\infty}(-1)^k\,\frac{(z+i)^k}{(1-i)^{k+1}}, \qquad R = \sqrt{2}$$

$$\frac{z-1}{3-z} = (z-1)\frac{1}{2-(z-1)} = \frac{z-1}{2}\frac{1}{1+\frac{z-1}{2}} = \frac{z-1}{2}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]$$

$$= \frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots = \sum_{k=1}^{\infty}\frac{(z-1)^k}{2^k}, \qquad R=2$$

$$\frac{1+z}{1-z} = -1+\frac{2}{1-z} = -1+\frac{2}{1-i-(z-i)} = -1+\frac{2}{1-i}\frac{1}{1-\frac{z-i}{1-i}}$$

$$= -1+\frac{2}{1-i}\left[1+\frac{z-i}{1-i}+\frac{(z-i)^2}{(1-i)^2}+\cdots\right] = -1+\frac{2}{1-i}+\frac{2(z-i)}{(1-i)^2}+\frac{2(z-i)^2}{(1-i)^3}+\cdots$$

$$= -1+\sum_{k=0}^{\infty}\frac{2(z-i)^k}{(1-i)^{k+1}}, \qquad R=\sqrt{2}$$

5. (a) Suppose the principal branch of the logarithm $f(z) = \operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ is expanded in a Taylor series with center $z_0 = -1 + i$. Explain why $R = 1$ is the radius of the largest circle centered at $z_0 = -1 + i$ within which f is analytic.

(b) Show that within the circle $|z - (-1 + i)| = 1$ the Taylor series for f is

$$\operatorname{Log} z = \frac{1}{2}\ln 2 + \frac{3\pi}{4} i - \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1+i}{2}\right)^k (z+1-i)^k$$

(c) Show that the radius of convergence for the power series in part (b) is $R = \sqrt{2}$. Explain why this does not contradict the result in part (a).

(a) The distance from z_0 to the branch cut is one unit.

(b) $f(-1+i) = \mathrm{Log}\,(-1+i) = \ln\sqrt{2} + \frac{3\pi}{4}i = \frac{1}{2}\ln 2 + \frac{3\pi}{4}i$

$$f'(-1+i) = \left.\frac{1}{z}\right|_{z=-1+i} = \frac{1}{-1+i} = -\frac{1+i}{2}$$

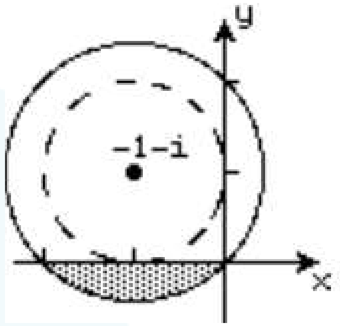
$$f''(-1+i) = -\left.\frac{1}{z^2}\right|_{z=-1+i} = -\left(\frac{1+i}{2}\right)^2$$

$$\vdots$$

$$f^{(k)}(-1+i) = -\left.\frac{(k-1)!}{z^k}\right|_{z=-1+i} = -(k-1)!\left(\frac{1+i}{2}\right)^2$$

$$\Rightarrow \mathrm{Log}\,z = \frac{1}{2}\ln 2 + \frac{3\pi}{4}i - \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1+i}{2}\right)^k (z+1-i)^k$$

(c) The series converges within the circle $|z+1-i| = \sqrt{2}$. Although the series converges in the shaded region, it does not converge to $\text{Log}\, z$ in this region.



6. (a) Consider the function $f(z) = \mathrm{Log}(1 + z)$. What is the radius of the largest circle centered at the origin within which f is analytic?

(b) Expand f in a Maclaurin series. What is the radius of convergence of this series?

(c) Use the result in part (b) to find a Maclaurin series for $\mathrm{Log}(1 - z)$.

(d) Find a Maclaurin series for $\mathrm{Log}\left(\dfrac{1+z}{1-z}\right)$

(a) $R = 1$, which is the distance from the origin to $z = -1$.

(b) Using Taylor's Theorem [or integrating the series for $1/(1 + z)$] we obtain for $R = 1$,

$$\mathrm{Log}(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k$$

(c) By replacing z in part (b) by $-z$ we obtain for $R = 1$,

$$\operatorname{Log}(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$$

(d) In general $\operatorname{Log}(z_1/z_2) \neq \operatorname{Log} z_1 - \operatorname{Log} z_2$ since $\operatorname{Log} z_1$ and $\operatorname{Log} z_2$ could differ by a constant multiple of i. That is, $\operatorname{Log} z_1 - \operatorname{Log} z_2 = Ci$ for some C. So

$$\operatorname{Log}\left(\frac{1+z}{1-z}\right) = \operatorname{Log}(1+z) - \operatorname{Log}(1-z) - Ci$$

When $z = 0$ we obtain $\operatorname{Log} 1 = \operatorname{Log} 1 - \operatorname{Log} 1 - Ci \Rightarrow C = 0$.

$$\operatorname{Log}\left(\frac{1+z}{1-z}\right) = 2z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \frac{2}{7}z^7 + \cdots = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)} z^{2k+1}, \qquad R = 1$$

7. Find a Maclaurin series for $\operatorname{erf}(z)$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \Rightarrow e^{-t^2} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{k!}$$

$$\frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^z t^{2k} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} z^{2k+1}$$

8. Expand $f(z) = \dfrac{z}{(z+1)(z-2)}$ in a Laurent series valid for the indicated annular domain

$0 < |z+1| < 3$ $\quad |z+1| > 3$ $\quad 1 < |z| < 2$ $\quad 0 < |z-2| < 3$

$$f(z) = \frac{z}{(z+1)(z-2)} = \frac{1/3}{z+1} + \frac{2/3}{z-2} = \frac{1}{3(z+1)} + \frac{2}{3}\frac{1}{-3+(z+1)}$$

$$= \frac{1}{3(z+1)} - \frac{2}{9}\frac{1}{1-\frac{(z+1)}{3}} = \frac{1}{3(z+1)} - \frac{2}{9}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \cdots\right]$$

$$= \frac{1}{3(z+1)} - \frac{2}{9} - \frac{2(z+1)}{3^3} - \frac{2(z+1)^2}{3^4} - \cdots$$

$0 < |z+1| < 3$

$$f(z)=\frac{z}{(z+1)(z-2)}=\frac{1}{3(z+1)}+\frac{2}{3}\frac{1}{(z+1)-3}=\frac{1}{3(z+1)}+\frac{2}{3(z+1)}\frac{1}{1-\frac{3}{z+1}}$$

$$\frac{1}{3(z+1)}+\frac{2}{3(z+1)}\left[1+\frac{3}{z+1}+\frac{3^2}{(z+1)^2}+\cdots\right]$$

$$=\frac{1}{z+1}+\frac{2}{(z+1)^2}+\frac{2\times 3}{(z+1)^3}+\frac{2\times 3^2}{(z+1)^4}+\cdots \qquad |z+1|>3$$

$$f(z)=\frac{1/3}{z+1}+\frac{2/3}{z-2}=\frac{1}{3z}\frac{1}{1+\frac{1}{z}}-\frac{1}{3}\frac{1}{1-\frac{z}{2}}$$

$$=\frac{1}{3z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right]-\frac{1}{3}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right]$$

$$=\cdots-\frac{1}{3z^2}+\frac{1}{3z}-\frac{1}{3}-\frac{z}{3\times 2}-\frac{z^2}{3\times 2^2}-\cdots$$

$1<|z|<2$

$$f(z)=\frac{2/3}{z-2}+\frac{1}{3}\frac{1}{3+z-2}=\frac{2/3}{z-2}+\frac{1}{9}\frac{1}{1-\frac{z-2}{3}}$$

$$=\frac{2/3}{z-2}+\frac{1}{9}\left[1+\frac{z-2}{3}+\frac{(z-2)^2}{3^2}+\cdots\right]$$

$$=\frac{2}{3(z-2)}+\frac{1}{9}+\frac{z-2}{3^3}+\frac{2(z-2)^2}{3^4}+\cdots \qquad 0<|z-2|<3$$

9. Determine the order of the poles for the given function

$$f(z) = \frac{\cot \pi z}{z^2} \qquad f(z) = \frac{1 - \cosh z}{z^4} \qquad f(z) = \frac{\sin z}{z^2 - z}$$

$f(z) = \dfrac{\cot \pi z}{z^2}$ $z = 0$ is a pole of order three. $z = \pm 1, \pm 2, \ldots$ are simple poles.

$$f(z) = \frac{1 - \cosh z}{z^4} = \frac{1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{6}{6!} + \cdots\right)}{z^4} = -\frac{1}{2! z^2} - \frac{1}{4!} - \frac{z^2}{6!} - \cdots$$

$z = 0$ is a pole of order two.

$$f(z) = \frac{\sin z}{z^2 - z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z(z-1)} = \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{z - 1}$$

$z = 1$ is a simple pole

1. Use Cauchy's residue theorem to evaluate the given integral along the indicated contour

$$\oint_C \frac{ze^z}{z^2-1}dz,\ C: |z| = 2 \qquad \oint_C \frac{\cot \pi z}{z^2}dz,\ C: |z| = \frac{1}{2}$$

$$\oint_C \frac{e^{iz} + \sin z}{(z-\pi)^4}dz,\ C: |z-3| = 1 \qquad \oint_C \frac{z}{(z+1)(z^2+1)}dz,\ C: \text{ellipse } 16x^2 + y^2 = 4$$

$$\oint_C \frac{2z-1}{z^2(z^3+1)}dz,\ C \text{ is the rectangle defined by } x = -2,\ y = -\frac{1}{2},\ y = 1$$

$$\oint_C \frac{ze^z}{z^2-1}dz = 2\pi i[Res(f(z), 1) + Res(f(z), -1)] = 2\pi i \cosh 1$$

$$\oint_C \frac{\cot \pi z}{z^2}dz = 2\pi i Res(f(z), 0)$$

$z = 0$ is a pole of order three $Res(f(z),\, 0) = \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} z\cot \pi z = -\frac{\pi}{3}$

$$\oint_C \frac{\cot \pi z}{z^2}\, dz = 2\pi i\left(-\frac{\pi}{3}\right) = -\frac{2\pi^2}{3} i$$

$$\oint_C \frac{e^{iz} + \sin z}{(z-\pi)^4}\, dz = Res(f(z),\, \pi) = \pi\left(-\frac{1}{3} + \frac{1}{3} i\right)$$

$$\oint_C \frac{z}{(z+1)(z^2+1)}\, dz = 2\pi i[Res(f(z),\, i) + Res(f(z),\, -i)]$$

$$= 2\pi i\left[\frac{1}{4} - \frac{1}{4} i + \frac{1}{4} + \frac{1}{4} i\right] = \pi i$$

$$\oint_C \frac{2z-1}{z^2(z^3+1)}\,dz = 2\pi i[Res(f(z),\,0) + Res(f(z),\,-1) + Res(f(z),\,\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i)]$$

$$= 2\pi i\left[2 + (-1) + (-\tfrac{1}{2} - \tfrac{1}{6}\sqrt{3}i)\right] = \pi\left(\frac{\sqrt{3}}{3} + i\right)$$

2. Evaluate the Cauchy principal value of the given improper integral

$$\int_0^{\infty} \frac{\cos 3x}{(x^2+1)^2}\,dx \qquad \int_0^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+1)^2}\,dx = 2\pi i Res(f(z), i) = 2\pi e^{-3}$$

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\,dx = Re\left(\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+1)^2}\,dx\right) = 2\pi e^{-3}$$

$$\int_0^{\infty} \frac{\cos 3x}{(x^2+1)^2}\,dx = \pi e^{-3}$$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)(x^2+4)}dx = 2\pi i[Res(f(z),\, i) + Res(f(z),\, 2i)] = \frac{\pi}{3}(e^{-1} - e^{-2})i$$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}dx = Im\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)(x^2+4)}dx\right) = \frac{\pi}{3}(e^{-1} - e^{-2})$$

$$\int_{0}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}dx = \frac{\pi}{6}(e^{-1} - e^{-2})$$

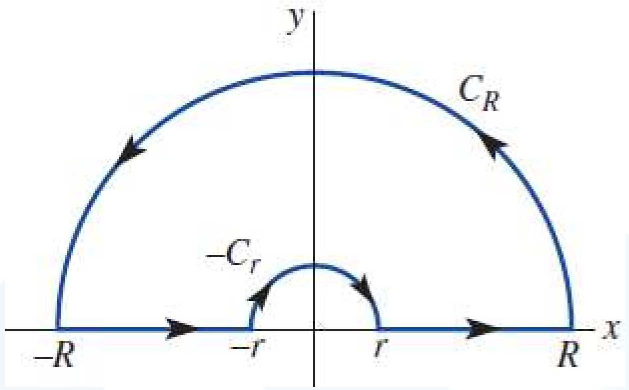
3. Evaluate the given trigonometric integral $\int_0^{2\pi} \frac{\cos^2\theta}{2+\sin\theta}\,d\theta$

$$\int_0^{2\pi} \frac{\cos^2\theta}{2+\sin\theta}\,d\theta = \frac{1}{2}\oint_C \frac{z^4+2z^2+1}{z^2(z^2+4iz-1)}\,dz,\ C: |z| = 1$$

$$= \pi i[Res(f(z),0) + Res(f(z),(-2+\sqrt{3})i)] = (4-2\sqrt{3})\pi$$

4. Use an indented contour and residues to establish the given result

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$



The function $f(z) = 1/z$ has simple poles at $z = 0$.

$$\oint_C \frac{e^{iz}}{z}\, dz = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_{r}^{R} = 0$$

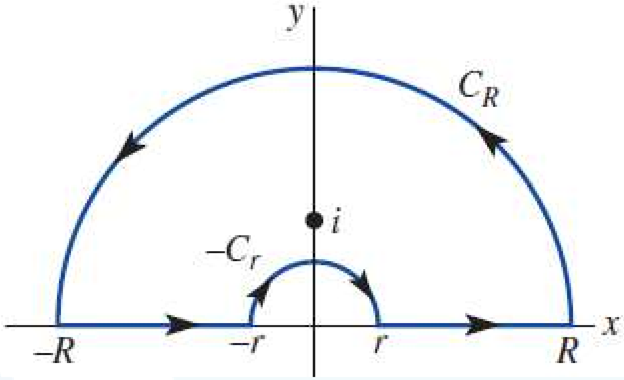
Taking the limits $R \to \infty$ and $r \to 0$, we find

$$P.V.\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - \pi i\, Res\,(f(z)e^{iz}, 0) = 0 \Rightarrow P.V.\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}\, dx = \pi i \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \pi(1-e^{-1})$$

The function $\dfrac{1}{z(z^2+1)}$ has simple poles at $z = 0,\ i$.



$$\oint_C \frac{e^{iz}}{z(z^2+1)}\,dz = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R \quad = 2\pi i\, Res\,(f(z)e^{iz}, i)$$

$R \to \infty$ and $r \to 0$,

$$P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,dx - \pi i\, Res\,(f(z)e^{iz}, 0) = 2\pi i\, Res\,(f(z)e^{iz}, i)$$

$$P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,dx = \pi i + 2\pi i\left(-\frac{e^{-1}}{2}\right) = \pi(1-e^{-1})i$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x(x^2+1)}\,dx = \pi(1-e^{-1})i \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \pi(1-e^{-1})$$

5. Establish the general result $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(\sqrt{a^2-1})^3}$, $a > 1$

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2}{i}\oint_C \frac{z}{(z^2+2az+1)^2}dz$$

$$= \frac{2}{i}\oint_C \frac{z}{(z-r_1)^2(z-r_2)^2}dz,\ C\colon |z| = 1,\ r_1 = -a+\sqrt{a^2-1},\ r_1 = -a-\sqrt{a^2-1}$$

$$\oint_C \frac{z}{(z-r_1)^2(z-r_2)^2}dz = 2\pi i(Res(f(z),\, r_1) = 2\pi i\frac{a}{4(\sqrt{a^2-1})^3} = \frac{a\pi}{2(\sqrt{a^2-1})^3}i$$

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2}{i}\frac{a\pi}{2(\sqrt{a^2-1})^3}i = \frac{a\pi}{(\sqrt{a^2-1})^3}$$

6. Establish the general result

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right), \ a > b > 0$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{i}{2b}\oint_C \frac{z^2 - 1}{z^2(z - r_1)(z - r_2)} dz, \ C: |z| = 1, \ r_{1,2} = \left(-a \pm \sqrt{a^2 - b^2}\right)/b$$

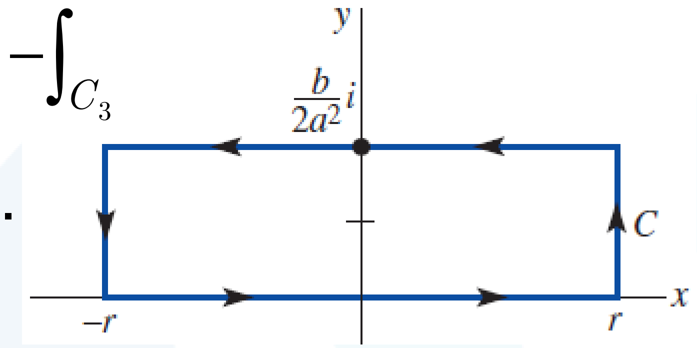
$$\oint_C \frac{z^2 - 1}{z^2(z - r_1)(z - r_2)} dz = 2\pi i\left[Res(f(z), 0) + Res(f(z), r_1)\right]$$

$$= 2\pi i\left[-\frac{2a}{b} + \frac{2\sqrt{a^2 - b^2}}{b}\right]$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right), \quad a > b > 0$$

7. Show that $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = e^{-b^2/4a^2}\sqrt{\pi}/2a$ by considering the complex integral $\oint_C e^{-a^2z^2}e^{ibz}dz$ along the contour C shown below. Use $\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \sqrt{\pi}/a$

$$\oint_C e^{-a^2z^2}e^{ibz}dz = \int_{-r}^{r} + \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 \Rightarrow \int_{-r}^{r} = -\int_{C_1} - \int_{C_2} - \int_{C_3}$$



C_1 and C_3 denote the vertical sides of the rectangle.

By the ML-inequality, $\int_{C_1} \to 0$ and $\int_{C_3} \to 0$ as $r \to \infty$

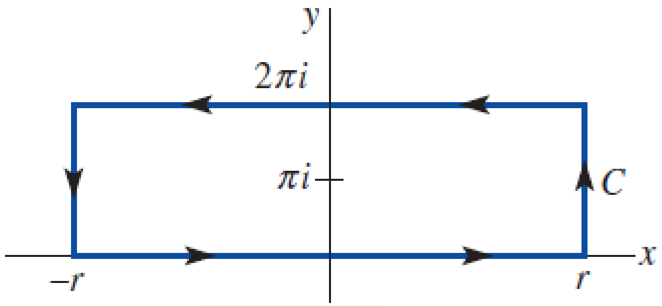
On C_2, $z = x + \frac{b}{2a^2}i, \quad -r \le x \le r,\ dz = dx$

$$\int_{-\infty}^{\infty} e^{-a^2x^2}e^{ibx}\,dx = -\int_{\infty}^{-\infty} e^{-a^2(x+\frac{b}{2a^2}i)^2}e^{ib(x+\frac{b}{2a^2}i)}dx = \int_{-\infty}^{\infty} e^{-a^2x^2}e^{-b^2/4a^2}\,dx$$

$$\int_{-\infty}^{\infty} e^{-a^2x^2}(\cos bx + i\sin bx)\,dx = e^{-b^2/4a^2}\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}e^{-b^2/4a^2}$$

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-b^2/4a^2}$$

8. Use the contour shown below to show that



$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin a\pi},\ 0<a<1$$

The function $\frac{e^{az}}{1+e^{z}}$ has simple poles at $z=\pi i$, $3\pi i$,$5\pi i$, ... in the upper plane.

$$\oint_C \frac{e^{az}}{1+e^{z}}\,dz = \int_{-r}^{r} + \int_{C_1} + \int_{C_2} + \int_{C_3} \quad = 2\pi i\, Res\,(f(z),\pi i) = -2\pi i e^{a\pi i}$$

On C_1, $z=r+iy$, $0 \le y \le 2\pi$, $dz = i\,dy$,

$$\left|\oint_{C_2} \frac{e^{az}}{1+e^{z}}\,dz\right| \le \frac{e^{ar}}{e^{r}-1} \times 2\pi \underset{r\to\infty}{\to} 0 \qquad (0<a<1)$$

On C_3, $z = -r + iy$, $0 \leq y \leq 2\pi$, $dz = i\,dy$

$$\left| \oint_{C_3} \frac{e^{az}}{1+e^{z}}\, dz \right| \leq \frac{e^{-ar}}{1-e^{-r}} \times 2\pi \underset{r\to\infty}{\to} 0 \qquad (0 < a)$$

On C_2, $z = x + 2\pi i$, $-r \leq x \leq r$, $dz = dx$

$$\oint_{C_2} \frac{e^{az}}{1+e^{z}}\, dz = \int_{-r}^{r} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\, dx = -e^{2a\pi i} \int_{-r}^{r} \frac{e^{ax}}{1+e^{x}}\, dx$$

as $r \to \infty$,

$$\int_{-r}^{r} \frac{e^{ax}}{1+e^{x}}\, dx - e^{2a\pi i} \int_{-r}^{r} \frac{e^{ax}}{1+e^{x}}\, dx = -2\pi i e^{a\pi i}$$

$$(1 - e^{2a\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\, dx = -2\pi i e^{a\pi i} \Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\, dx = \frac{\pi}{\sin a\pi}$$